

A CHARACTERIZATION OF TUTTE-COXETER GRAPH

A. MOHAMMADIAN AND M. FARROKHI D. G.

ABSTRACT. We give a natural generalization of the Tutte-Coxeter graph in a natural way and prove that the Tutte-Coxeter graph is the only vertex-transitive (edge-transitive) graph among all generalized Tutte-Coxeter graphs.

1. INTRODUCTION

In 1969, a class of generalized Petersen graphs was defined by Watkins [8] as follows: for positive integers $n \geq 3$ and $1 \leq k < n/2$, the generalized Petersen graph $P(n, k)$ is defined on a set of $2n$ vertices $u_0, u_1, \dots, u_{n-1}, v_0, v_1, \dots, v_{n-1}$ whose edges are $\{u_i, u_{i+1}\}$, $\{u_i, v_i\}$ and $\{v_i, v_{i+k}\}$, where i ranges over $\{1, 2, \dots, n\}$ and that all indices are taken modulo n . These graphs were studied earlier by Coxeter [3] and Robertson [7] in the special cases where n and k are relatively prime, and $k = 2$, respectively.

The automorphism group of generalized Petersen graphs are determined completely in [4]. It is shown that, with the exception of the dodecahedron $P(10, 2)$, the graph $P(n, k)$ is vertex-transitive if and only if $k^2 \equiv \pm 1 \pmod{n}$. Furthermore, $P(n, k)$ is a Cayley graph if and only if $k^2 \equiv 1 \pmod{n}$, see [5, 6]. Also, it is shown in [4] that $P(n, k)$ is edge-transitive if and only if

$$(n, k) \in \{(4, 1), (5, 2), (8, 3), (10, 2), (10, 3), (12, 5), (24, 5)\}.$$

For further results on generalized Petersen graphs we may refer the interested reader to Alspach [1], and Castagna and Prins [2].

The aim of this paper is to consider the same problem by defining a new class of cubic graphs arising from the Tutte-Coxeter graph. A *generalized Tutte-Coxeter graph* with respect to positive integers $n \geq 3$ (n even) and $1 \leq k < n/2$, denoted by $TC(n, k)$, is defined on a set of $3n$ vertices

$$a_0, a_1, \dots, a_{n-1}, b_0, b_1, \dots, b_{n-1}, c_0, c_1, \dots, c_{n-1}$$

whose its edges are

$$\{a_i, a_{i+1}\}, \{a_i, b_i\}, \{b_i, b_{i+n/2}\}, \{b_i, c_i\} \text{ and } \{c_i, c_{i+k}\},$$

where i ranges over $\{1, 2, \dots, n\}$. Here, all the indices are taken modulo n . We note that $T(10, 3)$ is the well-known Tutte-Coxeter graph. The same as for generalized Petersen graph, it is natural to ask which generalized Tutte-Coxeter graphs are vertex-transitive or edge-transitive? We shall prove that the only vertex-transitive (edge-transitive) graph among generalized Tutte-Coxeter graphs is the Tutte-Coxeter graph giving rise to a characterization of the Tutte-Coxeter graph.

2000 *Mathematics Subject Classification.* Primary 05C75; Secondary 05C30, 05E18.

Key words and phrases. Tutte-Coxeter graph, Petersen graph, vertex-transitive, edge-transitive, automorphism.

2. THE GENERALIZED TUTTE-COXETER GRAPH

We begin with naming the vertices and edges of the generalized Tutte-Coxeter graph as follows:

- The sets $\{a_0, a_1, \dots, a_{n-1}\}$, $\{b_0, b_1, \dots, b_{n-1}\}$ and $\{c_0, c_1, \dots, c_{n-1}\}$ of vertices denote the outer, middle and inner vertices, respectively.
- The sets $\{\{a_i, a_{i+1}\} : 1 \leq i \leq n\}$, $\{\{a_i, b_i\} : 1 \leq i \leq n\}$, $\{\{b_i, b_{i+n/2}\} : 1 \leq i \leq n\}$, $\{\{b_i, c_i\} : 1 \leq i \leq n\}$ and $\{\{c_i, c_{i+k}\} : 1 \leq i \leq n\}$ of edges denote the outer edges, spokes of type 1, middle edges, spokes of type 2 and inner edges, respectively.

It is easy to see that, the subgraph induced by inner vertices is a union of d disjoint (n/d) -cycles, where $d = \gcd(n, k)$.

For a given cycle C of $TC(n, k)$, let $ov(C)$, $mv(C)$ and $iv(C)$ be the number of outer, middle and inner vertices, and let $oe(C)$, $s_1e(C)$, $me(C)$, $s_2e(C)$ and $ie(C)$ be the number of outer edges, spokes of type 1, middle edges, spokes of type 2 and inner edges of C , respectively. Let \mathcal{C}_l be the set of all l -cycles of $TC(n, k)$ and put

$$\begin{aligned} OV(l) &= \sum_{C \in \mathcal{C}_l} ov(C), & OE(l) &= \sum_{C \in \mathcal{C}_l} oe(C), \\ MV(l) &= \sum_{C \in \mathcal{C}_l} mv(C), & S_1E(l) &= \sum_{C \in \mathcal{C}_l} s_1e(C), \\ IV(l) &= \sum_{C \in \mathcal{C}_l} iv(C), & ME(l) &= \sum_{C \in \mathcal{C}_l} me(C), \\ & & S_2E(l) &= \sum_{C \in \mathcal{C}_l} s_2e(C), \\ & & IE(l) &= \sum_{C \in \mathcal{C}_l} ie(C), \end{aligned}$$

for all $l \geq 3$. It is clear that the graph $TC(n, k)$ is vertex-transitive only if $OV(l) = MV(l) = IV(l)$ for all $l \geq 3$. Similarly, if the graph $TC(n, k)$ is edge-transitive, then we must have $OE(l) = S_1E(l) = 2ME(l) = S_2E(l) = IE(l)$ for all $l \geq 3$.

By evaluating the quantities $OV(l)$, $MV(l)$, $IV(l)$ and $OE(l)$, $S_1E(l)$, $ME(l)$, $S_2E(l)$, $IE(l)$, we will obtain all vertex-transitive and edge-transitive graphs among generalized Tutte-Coxeter graphs, respectively, when l takes only the values 6, 7 and 8.

First, we consider vertex-transitive graphs.

Theorem 2.1. *The graph $TC(n, k)$ is vertex-transitive if and only if $(n, k) = (10, 3)$.*

Proof. If $(n, k) = (10, 3)$, then $TC(n, k)$ is the Tutte-Coxeter graph and we are done. Now suppose that $(n, k) \neq (10, 3)$. Clearly, $TC(n, k)$ has 8-cycles of types (11) and (12) (see Table 1). If $k \geq 3$, then $TC(n, k)$ has 8-cycles of types different from (11) and (12) if and only if (n, k) is of the given forms in Table 2, in which all 8-cycles are described. In all cases, the equation $OV(8) = MV(8) = IV(8)$ is never satisfied so that $TC(n, k)$ is not vertex-transitive. Hence $k = 1$ or 2.

If $k = 1$ then by invoking Table 1, one can easily see that the equation $OV(8) = MV(8) = IV(8)$ holds only if $n = 8$. But then $T(8, 1)$ has only two families of different 7-cycles

$$\{a_i, a_{i+1}, a_{i+2}, a_{i+3}, a_{i+4}, b_{i+n/2}, b_i\}$$

and

$$\{c_i, c_{i+1}, c_{i+2}, c_{i+3}, c_{i+4}, b_{i+n/2}, b_i\}$$

for $i = 1, \dots, n$, from which it follows that $OV(7) = IV(7) = 5n$ and $MV(7) = 4n$. Hence, $TC(n, 1)$ is not vertex-transitive.

Finally, if $k = 2$ then one can easily see, by utilizing Table 3, that the equation $OV(7) = MV(7) = IV(7)$ is never satisfied so that $TC(n, 2)$ is not vertex-transitive. Then proof is complete. \square

Theorem 2.2. *The graph $TC(n, k)$ is edge-transitive if and only if $(n, k) = (10, 3)$.*

Proof. If $(n, k) = (10, 3)$, then $TC(n, k)$ is the Tutte-Coxeter graph and so is edge-transitive. Hence assume that $(n, k) \neq (10, 3)$. The same as in the proof of Theorem 2.1, it is easy to see that the equation $OE(8) = S_1E(8) = 2ME(8) = S_2E(8) = IE(8)$ is never satisfied when $k \geq 3$. Thus, we just consider the cases $k = 1$ and $k = 2$.

If $k = 1$ then the only possible 6-cycles in $TC(n, 1)$ are $\{a_i, a_{i+1}, b_{i+1}, c_{i+1}, c_i, b_i\}$ for $i = 1, \dots, n$ whenever $n \neq 6$. Hence $OE(6) = IE(6) = n$, $ME(6) = 0$ and $S_1E(6) = S_2E(6) = 2n$, which imply that $TC(n, 1)$ is not edge-transitive. In the case where $n = 6$, we have 8-cycles of types (1) , (9_+) , (9_-) , (10_+) , (10_-) , (11) and (12) as illustrated in Table 1. Therefore $OE(8) = IE(8) = 9n$, $ME(8) = 6n$ and $S_2E(8) = S_1E(8) = 4n$, and again $TC(n, 1)$ is not edge-transitive.

Finally, assume that $k = 2$. If $n \neq 6, 8, 14$ and 16 , then $TC(n, 2)$ has only the 7-cycles $\{a_i, a_{i+1}, a_{i+2}, b_{i+2}, c_{i+2}, c_i, b_i\}$ for $i = 1, \dots, n$. This shows that $OE(7) = S_1E(7) = S_2E(7) = 2n$, $ME(7) = 0$ and $IE(7) = n$, hence $TC(n, 2)$ is not edge-transitive. Invoking Table 3 in the cases where $n = 6, 8, 14$ or 16 , it yields that $OE(7) \neq IE(7)$, that is, $TC(n, 2)$ is not edge-transitive. The proof is complete. \square

REFERENCES

- [1] B. Alspach, The classification of Hamiltonian generalized Petersen graphs, *J. Combin. Theory Ser. B* **34** (1983), 293–312.
- [2] F. Castagna and G. Prins, Every generalized Petersen graph has a Tait colouring, *Pacific J. Math.* **40** (1972), 53–58.
- [3] H. S. M. Coxeter, Self-dual configurations and regular graphs, *Bull. Amer. Math. Soc.* **56** (1950), 413–455.
- [4] R. Frucht, J. E. Graver and M. E. Watkins, The groups of the generalized Petersen graphs, *Proc. Camb. Phil. Soc.* **70** (1971), 211–218.
- [5] M. Lovrečič-Saražin, A note on the generalized Petersen graphs that are also Cayley graphs, *J. Combin. Theory Ser. B* **69** (1997), 226–229.
- [6] R. Nedela and M. Škovič, Which generalized Petersen graphs are Cayley graphs, *J. Graph Theory* **19**(1) (1995), 1–11.
- [7] N. Robertson, *Graphs Minimal under Girth, Valency and Connectivity Constraints*, Ph.D. Thesis, University of Waterloo, Ontario, 1969.
- [8] M. E. Watkins, A theorem on Tait colorings with an application to generalized Petersen graphs, *J. Combin. Theory* **6** (1969), 152–164.

Table 1: $k \neq 2$

Type	8-Cycles	Conditions	#	ov	mv	iv	oe	s ₁ e	me	s ₂ e	ie
1	$\{a_i, a_{i+1}, a_{i+2}, b_{i+2}, c_{i+2}, c_{i+1}, c_i, b_i\}$	$k = 1$	n	3	2	3	2	2	0	2	2
2	$\{a_i, a_{i+1}, a_{i+2}, a_{i+3}, b_{i+3}, c_{i+3}, c_i, b_i\}$	$k = 3$ or $n - k = 3$	n	4	2	2	3	2	0	2	1
3	$\{a_i, a_{i+1}, b_{i+1}, c_{i+1}, c_{(i+1)+k}, c_{(i+1)+2k}, c_i, b_i\}$	$n = 3k + 1$	n	2	2	4	1	2	0	2	3
3'	$\{a_{i+1}, a_i, b_i, c_i, c_{i+k}, c_{i+2k}, c_{i+1}, b_{i+1}\}$	$n = 3k - 1$	n	2	2	4	1	2	0	2	3
4	$\{a_i, a_{i+1}, a_{i+2}, b_{i+2}, c_{i+2}, c_{(i+2)+k}, c_i, b_i\}$	$n = 2k + 2$	n	3	2	3	2	2	0	2	2
5	$\{a_i, a_{i+1}, a_{i+2}, a_{i+3}, a_{i+4}, a_{i+5}, a_{i+6}, a_{i+7}\}$	$n = 8$	1	8	0	0	8	0	0	0	0
6	$\{c_i, c_{i+k}, c_{i+2k}, c_{i+3k}, c_{i+4k}, c_{i+5k}, c_{i+6k}, c_{i+7k}\}$	$n = 8k$	k	0	0	8	0	0	0	0	8
6'	$\{c_i, c_{i+k}, c_{i+2k}, c_{i+3k}, c_{i+4k}, c_{i+5k}, c_{i+6k}, c_{i+7k}\}$	$3n = 8k$	$n/8$	0	0	8	0	0	0	0	8
7	$\{a_i, a_{i+1}, a_{i+2}, a_{i+3}, a_{i+4}, a_{i+5}, b_{i+5}, b_i\}$	$n = 10$	n	6	2	0	5	2	1	0	0
8	$\{b_i, c_i, c_{i+k}, c_{i+2k}, c_{i+3k}, c_{i+4k}, c_{i+5k}, b_{i+n/2}\}$	$n = 10k$	n	0	2	6	0	0	1	2	5
8'	$\{b_i, c_i, c_{i+k}, c_{i+2k}, c_{i+3k}, c_{i+4k}, c_{i+5k}, b_{i+n/2}\}$	$3n = 10k$	n	0	2	6	0	0	1	2	5
9 ₊	$\{a_i, a_{i+1}, a_{i+2}, b_{i+2}, c_{i+2}, c_{(i+2)+k}, b_{i+n/2}, b_i\}$	$n = 2k + 4$	n	3	3	2	2	2	1	2	1
9 ₋	$\{a_i, a_{-(i+1)}, a_{-(i+2)}, b_{-(i+2)}, c_{-(i+2)}, c_{-((i+2)+k)}, b_{i+n/2}, b_i\}$	$n = 2k + 4$	n	3	3	2	2	2	1	2	1
10 ₊	$\{a_i, a_{i+1}, b_{i+1}, c_{i+1}, c_{(i+1)+k}, c_{(i+1)+2k}, b_{i+n/2}, b_i\}$	$n = 4k + 2$	n	2	3	3	1	2	1	2	2
10 ₋	$\{a_i, a_{-(i+1)}, b_{-(i+1)}, c_{-(i+1)}, c_{-((i+1)+k)}, c_{-((i+1)+2k)}, b_{i+n/2}, b_i\}$	$n = 4k + 2$	n	2	3	3	1	2	1	2	2
10 ₊	$\{a_i, a_{i+1}, b_{i+1}, c_{i+1}, c_{((i+1)-k)}, c_{((i+1)-2k)}, b_{i+n/2}, b_i\}$	$n = 4k - 2$	n	2	3	3	1	2	1	2	2
10 ₋	$\{a_i, a_{-(i+1)}, b_{-(i+1)}, c_{-(i+1)}, c_{k-(i+1)}, c_{2k-(i+1)}, b_{i+n/2}, b_i\}$	$n = 4k - 2$	n	2	3	3	1	2	1	2	2
11	$\{a_i, a_{i+1}, b_{i+1}, b_{(i+1)+n/2}, a_{(i+1)+n/2}, a_{i+n/2}, b_{i+n/2}, b_i\}$	$n \geq 4$	$n/2$	4	4	0	2	4	2	0	0
12	$\{b_i, c_i, c_{i+k}, b_{i+k}, b_{(i+k)+n/2}, c_{(i+k)+n/2}, c_{i+n/2}, b_{i+n/2}\}$	$n \geq 4$	$n/2$	0	4	4	0	0	2	4	2

Table 2: $k \neq 1, 2$

The Values n	The Values k	Type of 8-circuits	OV	MV	IV	OE	S ₁ E	ME	S ₂ E	IE
$n = 3k + 1$	$k \neq 3$	(3), (11), (12)	4n	6n	6n	2n	4n	2n	4n	4n
$n = 3k - 1$	$k \neq 3, 5$	(3'), (11), (12)	4n	6n	6n	2n	4n	2n	4n	4n
$n = 2k + 2$	$k \neq 3$	(4), (11), (12)	5n	6n	5n	3n	4n	2n	4n	3n
$n = 8$	$k = 3$	(2), (3'), (4), (5), (6'), (11), (12)	12n	10n	12n	8n	8n	2n	8n	8n
$n = 8k$	$k \neq 3$	(6), (11), (12)	2n	4n	3n	n	2n	2n	2n	2n
$n = 8k$	$k = 3$	(2), (6), (11), (12)	6n	6n	5n	4n	4n	2n	4n	3n
$3n = 8k$	$k \neq 3, 6$	(6'), (11), (12)	2n	4n	3n	n	2n	2n	2n	2n
$n = 10$	$k = 4$	(4), (7), (11), (12)	11n	8n	5n	8n	6n	3n	4n	3n
$n = 10k$	$k \neq 3$	(8), (11), (12)	2n	6n	8n	n	2n	3n	4n	6n
$n = 10k$	$k = 3$	(2), (8), (11), (12)	6n	8n	10n	4n	4n	3n	6n	7n
$3n = 10k$	$k \neq 3$	(8'), (11), (12)	2n	6n	8n	n	2n	3n	4n	6n
$n = 2k + 4$	$k \neq 3, 5, 6$	(9 ₋), (9 ₊), (11), (12)	8n	10n	6n	5n	6n	4n	6n	3n
$n = 14$	$k = 5$	(3'), (9 ₋), (9 ₊), (11), (12)	10n	12n	10n	6n	8n	4n	8n	6n
$n = 16$	$k = 6$	(6'), (9 ₋), (9 ₊), (11), (12)	8n	10n	7n	5n	6n	4n	6n	4n
$n = 4k + 2$	$k \neq 3$	(10 ₋), (10 ₊), (11), (12)	6n	10n	8n	3n	6n	4n	6n	5n
$n = 4k + 2$	$k = 3$	(2), (10 ₋), (10 ₊), (11), (12)	10n	12n	10n	6n	8n	4n	8n	6n
$n = 4k - 2$	$k \neq 3$	(10 ₋ '), (10 ₊ '), (11), (12)	6n	10n	8n	3n	6n	4n	6n	5n

Table 3: $k = 2$

Type	7-Cycles	Conditions	#	ov	mv	iv	oe	s ₁ e	me	s ₂ e	ie
1	$\{a_i, a_{i+1}, a_{i+2}, b_{i+2}, c_{i+2}, c_i, b_i\}$	$n = 6, 8, 14$ or 16	n	3	2	2	2	2	0	2	1
2	$\{a_i, a_{i+1}, b_{i+1}, c_{i+1}, c_{i+3}, b_{i+3}, b_i\}$	$n = 6$	n	2	3	2	1	2	1	2	1
2'	$\{a_i, a_{-(i+1)}, b_{-(i+1)}, c_{-(i+1)}, c_{-(i+3)}, b_{i+3}, b_i\}$	$n = 6$	n	2	3	2	1	2	1	2	1
3	$\{a_i, a_{i+1}, a_{i+2}, a_{i+3}, a_{i+4}, b_{i+4}, b_i\}$	$n = 8$	n	5	2	0	4	2	1	0	0
4	$\{c_i, c_{i+2}, c_{i+4}, c_{i+6}, c_{i+8}, c_{i+10}, c_{i+12}\}$	$n = 14$	$n/7$	0	0	7	0	0	0	0	7
5	$\{c_{i+1}, c_{i+3}, c_{i+5}, c_{i+7}, c_{i+9}, c_{i+11}, c_{i+13}\}$	$n = 14$	$n/7$	0	0	7	0	0	0	0	7
6	$\{b_i, c_i, c_{i+2}, c_{i+4}, c_{i+6}, c_{i+8}, b_{i+8}\}$	$n = 16$	n	0	2	5	0	0	1	2	4

DEPARTMENT OF PURE MATHEMATICS, FERDOWSI UNIVERSITY OF MASHHAD, MASHHAD, IRAN
E-mail address: abbasmohammadian124@gmail.com

INSTITUTE FOR ADVANCED STUDIES IN BASIC SCIENCES (IASBS), AND THE CENTER FOR RE-
 SEARCH IN BASIC SCIENCES AND CONTEMPORARY TECHNOLOGIES, IASBS, P.O.Box 45195-1159,
 ZANJAN 66731-45137, IRAN

E-mail address: m.farrokhi.d.g@gmail.com, farrokhi@iasbs.ac.ir